Date: Mar. 31/14

UNIT 4 – DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

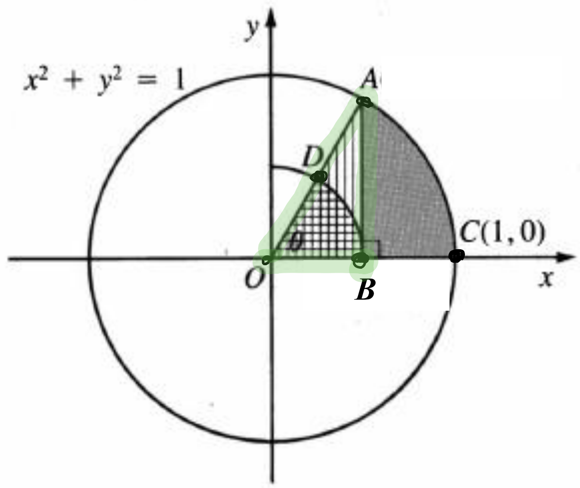
THE FUNDAMENTAL TRIGONOMETRIC LIMIT

Before we can find the derivatives of the trigonometric functions, we must find the *fundamental trigonometric limit*, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$, where θ is in radians.

θ (radians)	$\frac{\sin\theta}{\theta}$
0.5	$\doteq 0.958851$
0.2	$\doteq 0.993347$
0.01	$\doteq 0.999983$
0.001	$\doteq 0.9999998$

The trend of the values of $\frac{\sin\theta}{\theta}$ in the table suggests $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. A proof follows.

In the diagram, point A is on the unit circle $x^2 + y^2 = 1$. Point A determines angle θ, $0 < \theta < \frac{\pi}{2}$. The perpendicular drawn from point A meets the x-axis at B. The circle, radius OB, meets line segment OA at D.



In ΔOAB, $OA = 1$, find OB and BA.

Find OB

$\frac{OB}{OA} = \cos\theta$

$\frac{OB}{1} = \cos\theta$

$OB = \cos\theta$

Find BA

$\frac{BA}{OA} = \sin\theta$

$\frac{BA}{1} = \sin\theta$

$BA = \sin\theta$

Now Area sector $OBD \leqslant$ Area triangle $OAB \leqslant$ Area sector OAC

Therefore $\frac{1}{2}(OB)^2(\theta) \leqslant \frac{1}{2}(OB)(BA) \leqslant \frac{1}{2}(OC)^2(\theta)$

Sector Area

$= \frac{1}{2} r^2 \theta$

Area of Triangle

$= \frac{1}{2} bh$

$\frac{1}{2}(\cos\theta)^2(\theta) \le \frac{1}{2}(\cos\theta)(\sin\theta) \le \frac{1}{2}(1)^2\theta$

$\frac{1}{2}\theta \cdot \cos^2\theta \le \frac{1}{2}\cos\theta \cdot \sin\theta \le \frac{1}{2}\theta$

mult. by

$\theta\cos^2\theta \le \cos\theta \cdot \sin\theta \le \theta$

$\div$ by $\theta \cdot \cos\theta$

$\frac{\theta\cos^2\theta}{\theta \cdot \cos\theta} \le \frac{\cos\theta\sin\theta}{\theta \cdot \cos\theta} \le \frac{\theta}{\theta\cos\theta}$

$\cos\theta \le \frac{\sin\theta}{\theta} \le \frac{1}{\cos\theta}$

By the inequality theorem

$\lim_{\theta \to 0} \cos\theta \le \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \le \lim_{\theta \to 0} \frac{1}{\cos\theta}$

$\cos 0 \le \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \le \frac{1}{\cos 0}$

$1 \le \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \le 1$

By the squeeze theorem.

$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

The fundamental trigonometric limit,

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \quad \textit{or} \quad \lim_{h\to 0}\frac{\sin h}{h}=1$$

Ex. 1. Evaluate the following limits using the ***fundamental trig limit***, if the limit is ***indeterminate***.

a) $\displaystyle\lim_{h\to 0}\frac{\cos h-1}{h}$ $\quad \frac{0}{0}$

$$=\lim_{h\to 0}\frac{\cos h-1}{h}\cdot\frac{\cos h+1}{\cos h+1}$$

$$=\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}$$

$$=\lim_{h\to 0}\frac{-(1-\cos^2 h)}{h(\cos h+1)}$$

$$=\lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)}$$

$$=\lim_{h\to 0}\left[\frac{\sin h}{h}\cdot\frac{-\sin h}{\cos h+1}\right]$$

$$=\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{-\sin h}{\cos h+1}$$

$$=1\cdot\frac{-\sin 0}{\cos 0+1}$$

$$=1\cdot\frac{0}{2}$$

$$=0$$

b) $\displaystyle\lim_{5x\to 0}\frac{\sin 5x}{5x}$ $\quad$ Let $u=5x$

$$=\lim_{u\to 0}\frac{\sin u}{u}$$

$$=1$$

c) $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

$$=\lim_{\theta\to 0}\frac{\frac{\sin\theta}{\cos\theta}}{\frac{\theta}{1}}$$

$$=\lim_{\theta\to 0}\left[\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\theta}\right]$$

$$=\lim_{\theta\to 0}\left[\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}\right]$$

$$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to}\frac{1}{\cos\theta}$$

$$=(1)\cdot(1)$$

$$=1$$

d) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x}$

$$=\lim_{x\to 0}\frac{2\sin x\cdot\cos x}{x}$$

$$=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{2\cos x}{1}\right]$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}(2\cos x)$$

$$=(1)\cdot(2)$$

$$=2$$

or

d) $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{x}\cdot\frac{2}{2}$

$$=\lim_{x\to 0}\frac{2\cdot\sin 2x}{2x}$$

$$=2\lim_{x\to 0}\frac{\sin 2x}{2x}$$

$$=2\boxed{\lim_{2x\to 0}\frac{\sin 2x}{2x}}$$

$$=2(1)$$

$$=2$$

Note: If $x\to 0$ then $2x\to 0$

e) $\displaystyle\lim_{x\to 0}\frac{\sin 6x}{x}\cdot\frac{6}{6}$

$$=\lim_{x\to 0}\left[6\,\frac{\sin 6x}{6x}\right]$$

$$=6\cdot\lim_{x\to 0}\frac{\sin 6x}{6x}$$

$$=6\boxed{\lim_{6x\to 0}\frac{\sin 6x}{6x}}$$

$$=6(1)$$

$$=6$$

f) $\displaystyle\lim_{\theta\to 0}\left[\frac{1-\cos 4\theta}{\theta^2}\right]\left[\frac{1+\cos 4\theta}{1+\cos 4\theta}\right]$

$$=\lim_{\theta\to 0}\frac{1-\cos^2 4\theta}{\theta^2(1+\cos 4\theta)}$$

$$=\lim_{\theta\to 0}\frac{\sin^2 4\theta}{\theta^2(1+\cos 4\theta)}$$

$$=\lim_{\theta\to 0}\left[\frac{\sin 4\theta}{\theta}\cdot\frac{\sin 4\theta}{\theta}\cdot\frac{1}{1+\cos 4\theta}\right]$$

$$=\lim_{\theta\to 0}\frac{\sin 4\theta}{\theta}\cdot\frac{4}{4}\cdot\lim_{\theta\to 0}\frac{\sin 4\theta}{\theta}\cdot\frac{4}{4}\lim_{\theta\to 0}\frac{1}{1+\cos 4\theta}$$

$$=4\lim_{4\theta\to 0}\frac{\sin 4\theta}{4\theta}\cdot 4\lim_{4\theta\to 0}\frac{\sin 4\theta}{4\theta}\cdot\frac{1}{2}$$

$$=4(1)\cdot 4(1)\cdot\frac{1}{2}$$

$$=8$$

g) $\displaystyle\lim_{x\to 0}\frac{\tan(\sin x)}{\sec x}$

$$=\frac{\tan(\sin 0)}{\sec 0}$$

$$=\frac{\tan(0)}{1}$$

$$=0$$

h) $\displaystyle\lim_{x\to\pi}\frac{\sin x}{x-\pi}$ $\quad \frac{0}{0}$

Let $x-\pi=\theta$
If $x\to\pi$,
$\theta\to\pi-\pi$
$\theta\to 0$

If $x-\pi=\theta$
$x=\theta+\pi$

$$=\lim_{\theta\to 0}\frac{\sin(\theta+\pi)}{\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta\cdot\cos\pi+\sin\pi\cos\theta}{\theta}$$

$$=\lim_{\theta\to 0}\frac{\sin\theta(-1)+(0)\cdot\cos\theta}{\theta}$$

$$=\lim_{\theta\to 0}\left[\frac{-\sin\theta}{\theta}\right]$$

$$=-1\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$$

$$=-1(1)$$

$$=-1$$

HW: Worksheet on Trig Limits

Date: April 2/14

The Derivatives of sin*x* and cos*x*

Recall:

Fundamental Trig Limits

1. $\lim\limits_{h\to 0}\frac{\sin h}{h}$ $= 1$

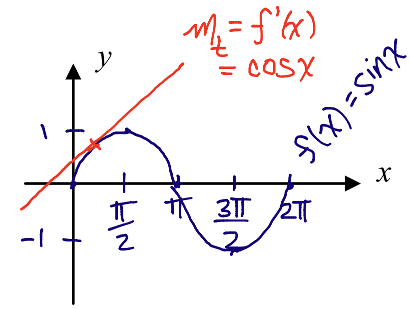
2. $\lim\limits_{h\to 0}\frac{\cos h - 1}{h}$ $= 0$

Compound Angle Formulas

1. $\sin(A+B) = \sin A\cdot\cos B + \cos A\cdot\sin B$

2. $\cos(A+B) = \cos A\cdot\cos B - \sin A\cdot\sin B$

Ex. 1. Find the derivative of $f(x) = \sin x$ from ***first principles***.



$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\cdot\sin h}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\cdot\sin h}{h}\right]$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0}\frac{\cos x\cdot\sin h}{h}$$

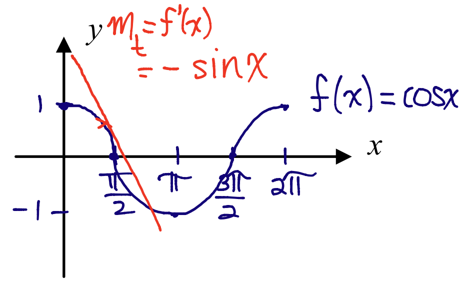
$$= \lim_{h\to 0}\left[\sin x\cdot\frac{\cos h - 1}{h}\right] + \lim_{h\to 0}\left[\cos x\cdot\frac{\sin h}{h}\right]$$

$$= \lim_{h\to 0}\sin x\cdot\boxed{\lim_{h\to 0}\frac{\cos h - 1}{h}} + \lim_{h\to 0}\cos x\cdot\boxed{\lim_{h\to 0}\frac{\sin h}{h}}$$

$$= \sin x\,(0) + \cos x\,(1)$$

$$= \cos x$$

Ex. 2. Find the derivative of $f(x) = \cos x$ from ***first principles***.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cdot \cos h - \sin x \cdot \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x(\cos h - 1)}{h} - \frac{\sin x \cdot \sin h}{h} \right]$$

$$= \lim_{h \to 0} \left[\cos x \cdot \frac{\cos h - 1}{h} \right] - \lim_{h \to 0} \left[\sin x \cdot \frac{\sin h}{h} \right]$$

$$= \lim_{h \to 0} \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}} - \lim_{h \to 0} \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}$$

$$= \cos x\,(0) - \sin x\,(1)$$

$$= -\sin x$$

$\therefore$ If $f(x) = \cos x$ then $f'(x) = -\sin x$

SUMMARY

If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

If $y = \cos x$ then $\frac{dy}{dx} = -\sin x$

If $y = \sin f(x)$ then $\frac{dy}{dx} = \cos f(x) \cdot f'(x)$

If $y = \cos f(x)$ then $\frac{dy}{dx} = -\sin f(x) \cdot f'(x)$

If $y = [\sin f(x)]^n$ then $\cancel{\frac{dy}{dx}} =$

$$\frac{dy}{dx} = n[\sin f(x)]^{n-1} \cdot \cos f(x) \cdot f'(x)$$

If $y = [\cos f(x)]^n$ then $\cancel{\frac{dy}{dx}} =$

$$\frac{dy}{dx} = n[\cos f(x)]^{n-1} \cdot [-\sin f(x)] \cdot f'(x)$$

Ex. 3. Find $\frac{dy}{dx}$.

a) $y=\cos x^2$

$\frac{dy}{dx}=-\sin x^2\cdot 2x$

$\therefore \frac{dy}{dx}=-2x\cdot\sin x^2$

b) $y=\cos^2 x$

$y=(\cos x)^2$

$\frac{dy}{dx}=2(\cos x)^1\cdot(-\sin x)\cdot 1$

$\frac{dy}{dx}=-2\sin x\cdot\cos x$

or

$\frac{dy}{dx}=-\sin 2x$

c) $y=\sin\frac{1}{x}$

$y=\sin x^{-1}$

$\frac{dy}{dx}=+\cos x^{-1}\cdot(-x^{-2})$

$\therefore \frac{dy}{dx}=-\frac{1}{x^2}\cdot\cos\frac{1}{x}$

d) $y=\sin^3(3\pi-x)$

$y=[\sin(3\pi-x)]^3$

$\frac{dy}{dx}=3[\sin(3\pi-x)]^2\cdot\cos(3\pi-x)\cdot(-1)$

$\frac{dy}{dx}=-3\sin^2(3\pi-x)\cdot\cos(3\pi-x)$

e) $y=\sin 4x\cdot\cos 2x$

$\frac{dy}{dx}=\cos 4x\cdot 4\cdot\cos 2x+[-\sin 2x\cdot 2]\cdot\sin 4x$

$\therefore \frac{dy}{dx}=4\cos 4x\cdot\cos 2x-2\sin 2x\cdot\sin 4x$

f) $y=\frac{\sin x}{1-2\cos x}$

$\frac{dy}{dx}=\frac{\cos x(1-2\cos x)-[-2(-\sin x\cdot 1)]\sin x}{(1-2\cos x)^2}$

$=\frac{\cos x-2\cos^2 x-2\sin^2 x}{(1-2\cos x)^2}$

$=\frac{\cos x-2(\cos^2 x+\sin^2 x)}{(1-2\cos x)^2}$

$\therefore \frac{dy}{dx}=\frac{\cos x-2}{(1-2\cos)^2}$

g) $\sin xy+3y=0$

diff. w.r.t. x

$\cos xy\left[1\cdot y+\frac{dy}{dx}\cdot x\right]+3\frac{dy}{dx}=0$

$y\cdot\cos xy+x\cos xy\frac{dy}{dx}+3\frac{dy}{dx}=0$

$\frac{dy}{dx}(x\cdot\cos xy+3)=-y\cos xy$

$\therefore \frac{dy}{dx}=\frac{-y\cdot\cos xy}{x\cos xy+3}$

h) $y=\sin(\cos 4x)$

$\frac{dy}{dx}=\cos(\cos 4x)\cdot[-\sin 4x\cdot 4]$

$\frac{dy}{dx}=-4\sin 4x\cdot\cos(\cos 4x)$

Ex. 4. Find the equation of the tangent to the curve $y\sin 2x = y\cos 2x$ at $\left(\frac{\pi}{4}, 1\right)$.

$$y \cdot \sin 2x = y \cdot \cos 2x$$

diff. w.r.t. x

$$\frac{dy}{dx} \cdot \sin 2x + \underbrace{\cos 2x \cdot 2}\cdot y = \frac{dy}{dx} \cdot \cos 2x + \underbrace{(-\sin 2x \cdot 2)}\cdot y$$

$$\frac{dy}{dx}\sin 2x + 2y\cos 2x = \frac{dy}{dx}\cos 2x - 2y \cdot \sin 2x$$

Find $\frac{dy}{dx}$ if $x = \frac{\pi}{4}$ & 1

$$\frac{dy}{dx} \cdot \sin\frac{\pi}{2} + 2\cos\frac{\pi}{2} = \frac{dy}{dx} \cdot \cos\frac{\pi}{2} - 2\sin\frac{\pi}{2}$$

$$\frac{dy}{dx}(1) + 2(0) = \frac{dy}{dx}(0) - 2(1)$$

$$\frac{dy}{dx} = -2$$

For tangent,

$m_t = -2$; $\left(\frac{\pi}{4}, 1\right)$; $b =$ ___

$$1 = -2\left(\frac{\pi}{4}\right) + b$$

$$1 = -\frac{\pi}{2} + b$$

$$1 + \frac{\pi}{2} = b$$

$$\frac{2+\pi}{2} = b$$

$\therefore$ the equation of the tangent to the curve at $\left(\frac{\pi}{4}, 1\right)$ is

$$y = -2x + \frac{2+\pi}{2} \text{ or}$$

$$4x + 2y - 2 - \pi = 0$$

HW: Worksheet on Derivatives of sin *x* and cos *x*: #1, 3, 4ac, 6, 8, 9 (equation of tangent only)