Date: April 3/14

DERIVATIVES OF THE OTHER TRIG FUNCTIONS

Ex. 1. Find $\frac{dy}{dx}$ for each of the following reciprocal trig functions and illustrate graphically.

a) $y = \csc x$

$y = \frac{1}{\sin x}$

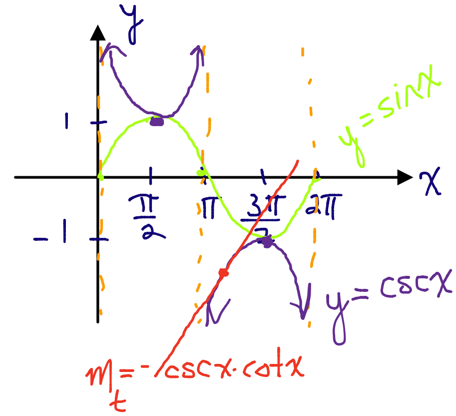
$y = (\sin x)^{-1}$

$\frac{dy}{dx} = -(\sin x)^{-2} \cdot \cos x \cdot 1$

$= -\frac{\cos x}{\sin^2 x}$

$= -\boxed{\frac{1}{\sin x}} \cdot \boxed{\frac{\cos x}{\sin x}}$

$\therefore \frac{dy}{dx} = -\csc x \cdot \cot x$



b) $y = \sec x$

$y = \frac{1}{\cos x}$

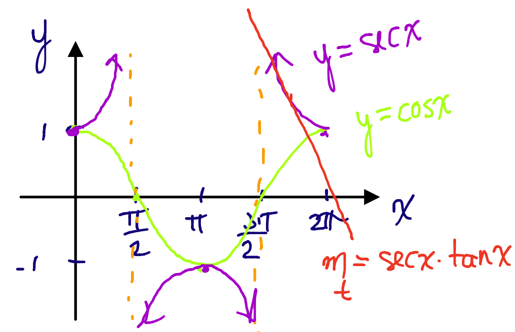
$y = (\cos x)^{-1}$

$\frac{dy}{dx} = -(\cos x)^{-2}(-\sin x \cdot 1)$

$= +\frac{\sin x}{\cos^2 x}$

$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$

$\frac{dy}{dx} = \sec x \cdot \tan x$



c) $y = \tan x$

$y = \frac{\sin x}{\cos x}$

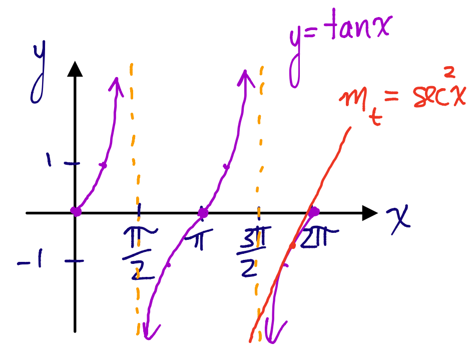
$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - (-\sin x) \cdot \sin x}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \frac{1}{\cos^2 x}$

$= \left(\frac{1}{\cos x}\right)^2$

$\therefore \frac{dy}{dx} = \sec^2 x$



d) $y = \cot x$

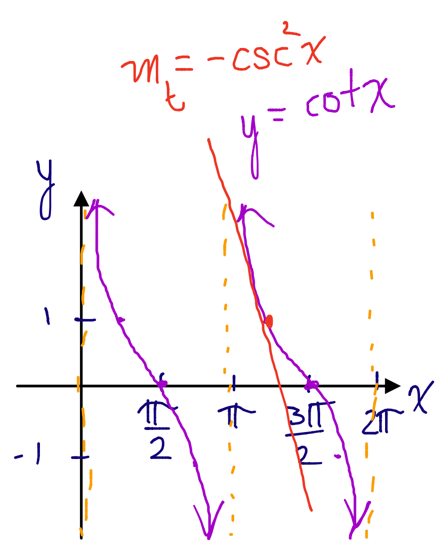
$y = \frac{\cos x}{\sin x}$

$\frac{dy}{dx} = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{(\sin x)^2}$

$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$

$= \frac{-1}{\sin^2 x}$

$\therefore \frac{dy}{dx} = -\csc^2 x$



SUMMARY OF TRIGONOMETRIC DERIVATIVES

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Ex. 2. Find $\frac{dy}{dx}$ for the following:

a) $y = \tan(x^3 + 4x)$

$$\frac{dy}{dx} = \sec^2(x^3+4x)\,(3x^2+4)$$

$$\frac{dy}{dx} = (3x^2+4)\cdot\sec^2(x^3+4x)$$

b) $y = 2x^3 \cdot \cot\left(\frac{1}{x}\right)$

x^{-1}

$$\frac{dy}{dx} = 6x^2\cdot\cot\left(\frac{1}{x}\right) + \left[-\csc^2\left(\frac{1}{x}\right)\cdot(-x^{-2})\right]\cdot 2x^3$$

$$\frac{dy}{dx} = 6x^2\cdot\cot\left(\frac{1}{x}\right) + 2x^1\cdot\csc^2\left(\frac{1}{x}\right)$$

c) $y = \csc(\tan x)$

$$\frac{dy}{dx} = -\csc(\tan x)\cdot\cot(\tan x)\cdot\sec^2 x$$

$$\frac{dy}{dx} = -\sec^2 x\cdot\csc(\tan x)\cdot\cot(\tan x)$$

d) $y = \sec^3 \pi x$

$$y = [\sec\pi x]^3$$

$$\frac{dy}{dx} = 3[\sec\pi x]^2\cdot\sec\pi x\cdot\tan\pi x\cdot\pi$$

$$\frac{dy}{dx} = 3\pi\cdot\tan\pi x\cdot\sec^3\pi x$$

e) $\cot(x + y) = 1 - y$

diff. w.r.t. x

$$-\csc^2(x+y)\left[1+\frac{dy}{dx}\right] = -\frac{dy}{dx}$$

$$-\csc^2(x+y) - \csc^2(x+y)\cdot\frac{dy}{dx} = -\frac{dy}{dx}$$

$$\frac{dy}{dx}\left[1-\csc^2(x+y)\right] = \csc^2(x+y)$$

$$\frac{dy}{dx} = \frac{\csc^2(x+y)}{1-\csc^2(x+y)}$$

$$\frac{dy}{dx} = \frac{\frac{1}{\sin^2(x+y)}}{1-\frac{1}{\sin^2(x+y)}}\cdot\frac{\sin^2(x+y)}{\sin^2(x+y)}$$

$$= \frac{1}{\sin^2(x+y)-1}\cdot\frac{-1}{-1}$$

$$= \frac{-1}{1-\sin^2(x+y)}$$

$$= \frac{-1}{\cos^2(x+y)}$$

$$\therefore \frac{dy}{dx} = -\sec^2(x+y)$$

H.W. Worksheet on Further Trigonometric Derivatives #1, 4, 5

Date: April. 7

APPLICATIONS OF TRIG FUNCTIONS
RELATED RATES

Ex. 1. The motion of a particle is described by the function $s(t) = 10\cos\left(5t - \frac{\pi}{4}\right)$. Find the velocity and acceleration functions and determine the maximum and minimum values of each.

$s(t) = 10\cos\left(5t - \frac{\pi}{4}\right)$

$v(t) = s'(t)$

$= 10\left[-\sin\left(5t - \frac{\pi}{4}\right)\right] \cdot 5$

$\therefore v(t) = -50\sin\left(5t - \frac{\pi}{4}\right)$

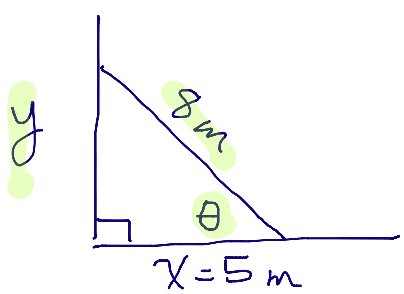
$a(t) = v'(t)$

$= -50\left[+\cos\left(5t - \frac{\pi}{4}\right)\right] \cdot 5$

$\therefore a(t) = -250\cos\left(5t - \frac{\pi}{4}\right)$

The maximum and minimum velocities are 50 and -50. The maximum and minimum accelerations are 250 and -250.

Ex. 2. A ladder of length 8 m leaning against a wall starts to slide. If its upper end slides down the wall at a rate of 0.25 m/s, at what rate is the angle between the ladder and the ground changing when the foot of the ladder is 5 m from the wall?



Given: $\frac{dy}{dt} = -0.25$ m/s

Find: $\frac{d\theta}{dt}$ when $x = 5$ m.

Find $\cos\theta$ if $x = 5$

$\cos\theta = \frac{5}{8}$

$\sin\theta = \frac{y}{8}$

$8\sin\theta = y$

diff. w.r.t "t"

$8 \cdot \cos\theta \boxed{\frac{d\theta}{dt}} = \frac{dy}{dt}$

$8\left(\frac{5}{8}\right)\frac{d\theta}{dt} = -\frac{1}{4}$

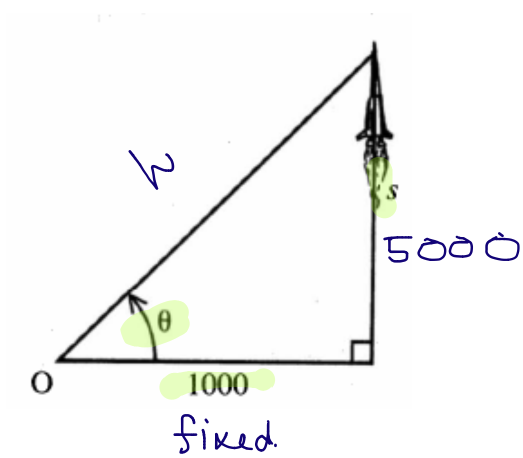
$\frac{d\theta}{dt} = -\frac{1}{20}$

$\frac{1}{20}$ rad/s.

$= \frac{1}{20} \times \frac{180}{\pi}$ °/s.

$\therefore$ the angle between the ladder & the ground is decreasing at an exact rate of $\frac{1}{20}$ rad/s or an approx. rate of 2.86 °/s.

Ex. 3. A TV camera, located 1000 m from the site of a rocket launch, records the event. The rocket is launched vertically, and its elevation s (in metres) t seconds after lift-off is $s = 200t^2$. How rapidly must the camera angle be increased in order to maintain a view of the rocket, 5 s after lift-off?



Find $\frac{d\theta}{dt}$ when $t=5$ s.

$$\tan\theta = \frac{s}{1000}$$

$\because$ $s = 200t^2$

$$\tan\theta = \frac{200t^2}{1000}$$

$$\tan\theta = \frac{t^2}{5}$$

$$5\tan\theta = t^2$$

diff. w.r.t. "t"

$$5 \cdot \sec^2\theta \cdot \frac{d\theta}{dt} = 2t$$

Find $\sec^2\theta$ if $t=5$

$$s = 200(5)^2$$
$$= 5000$$

Find. h^2

$$h^2 = 1000^2 + 5000^2$$
$$h^2 = 26\,000\,000$$
$$\sec^2\theta = \frac{26\,000\,000}{1\,000\,000}$$
$$\sec^2\theta = 26$$

$$\cancel{5}(26)\frac{d\theta}{dt} = 2(\cancel{5})$$

$$\frac{d\theta}{dt} = \frac{1}{13}$$

$\therefore$ the camera angle must increase at an exact rate of $\frac{1}{13}$ radian/s or an approx. rate of $4.41°/s$.

– Related Rates

HW: Worksheet on Applications of Trig Functions ~~Part A #5abde; Part B #1 to 6, 11 to 13~~

Date: April 8/14

APPLICATIONS OF TRIG FUNCTIONS
OPTIMIZATION

Ex. 1. Find the maximum and minimum values of $f(x) = \cos 2x + 2\sin x$ for $0 \le x \le \frac{3\pi}{4}$.

$f(x) = \cos 2x + 2\sin x$

$f'(x) = -2\sin 2x + 2\cos x$

For max/min

$f'(x) = 0$

$-2\sin 2x + 2\cos x = 0$

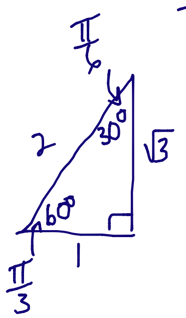
÷ by -2: $\sin 2x - \cos x = 0$

$2\sin x \cdot \cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\therefore \cos x = 0$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ in QI $x = \frac{\pi}{6}$ in QII $\cancel{x = \frac{5\pi}{6}}$



* $0 \le x \le \frac{3\pi}{4}$

x	$f(x)$	
0	1	← min
$\frac{\pi}{6}$	$1\frac{1}{2}$	max
$\frac{\pi}{2}$	1	
$\frac{3\pi}{4}$	$\sqrt{2}$	

$f\left(\frac{\pi}{2}\right) = \cos\pi + 2\sin\frac{\pi}{2}$
$= -1 + 2(1)$
$= 1$

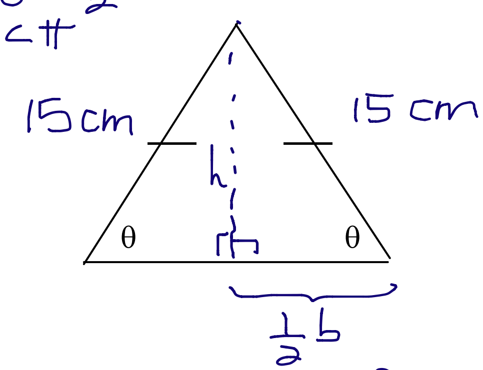
$f\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{2} + 2\cdot\sin\frac{3\pi}{4}$
$= 0 + 2\left(+\frac{1}{\sqrt{2}}\right)$
$= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2}$

∴ the maximum value of $f(x)$ is $1\frac{1}{2}$ and the minimum value of $f(x)$ is 1.

Ex. 2. An isosceles triangle has two equal sides of length 15 cm. What is the maximum possible area of the triangle?

$0 < \theta < \frac{\pi}{2}$

$0 < 2\theta < \pi$



* Find $\frac{1}{2}b$ in terms of θ

$\frac{\cos\theta}{1} = \frac{\frac{1}{2}b}{15}$

$\frac{1}{2}b = 15\cos\theta$

* Find h in terms of θ

$\frac{\sin\theta}{1} = \frac{h}{15}$

$h = 15\sin\theta$

Maximize the area, A, in cm^2

$A = \frac{1}{2}bh$

$= 15\cos\theta \cdot 15\sin\theta$

$A = 225\sin\theta\cos\theta \cdot \frac{2}{2}$

$A = \frac{225}{2} \cdot 2\sin\theta\cos\theta$

$A = \frac{225}{2} \cdot \sin 2\theta$

$\frac{dA}{d\theta} = \frac{225}{\cancel{2}} \cdot \cos 2\theta \cdot \cancel{2}$

For max/min, $\frac{dA}{d\theta} = 0$

$0 = 225\cos 2\theta$

$\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{2} \times \frac{1}{2}$

$\theta = \frac{\pi}{4}$

if $\theta = \frac{\pi}{4}$

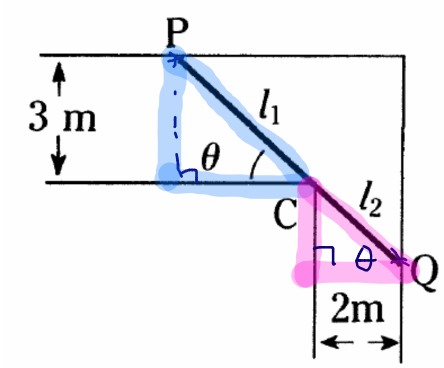
$A = \frac{225}{2} \cdot \sin\frac{\pi}{2}$

$= \frac{225}{2} \cdot 1$

$= 112\frac{1}{2}$

∴ maximum possible area is $112\frac{1}{2}$ cm^2.

Ex. 3. Two corridors, 3 m and 2 m wide respectively, meet at right angles. Find the length of the longest thin straight rod that can pass horizontally around the corner. Neglect the thickness of the rod. Answer to the nearest cm.



Find l_1 & l_2 in terms of θ

$\frac{\sin\theta}{1} = \frac{3}{l_1}$ $\qquad$ $\frac{\cos\theta}{1} = \frac{2}{l_2}$

$l_1 \cdot \sin\theta = 3$ $\qquad$ $l_2 \cdot \cos\theta = 2$

$l_1 = \frac{3}{\sin\theta}$ $\qquad$ $l_2 = \frac{2}{\cos\theta}$

maximize the length, l, of the rod in m

$l = l_1 + l_2$

$l = \frac{3}{\sin\theta} + \frac{2}{\cos\theta}$ *

$l = 3(\sin\theta)^{-1} + 2(\cos\theta)^{-1}$

$\frac{dl}{d\theta} = -3(\sin\theta)^{-2} \cdot \cos\theta - 2(\cos\theta)^{-2}(-\sin\theta)$

$\frac{dl}{d\theta} = -\frac{3\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta}$

For max/min, $\frac{dl}{d\theta} = 0$

$-\frac{3\cos\theta}{\sin^2\theta} + \frac{2\sin\theta}{\cos^2\theta} = 0$

$\frac{2\sin\theta}{\cos^2\theta} = \frac{3\cos\theta}{\sin^2\theta}$

$2\sin^3\theta = 3\cos^3\theta$

$\frac{\cancel{2}\sin^3\theta}{\cancel{2}\cos^3\theta} = \frac{3\cancel{\cos^3\theta}}{2\cancel{\cos^3\theta}}$

$\tan^3\theta = \frac{3}{2}$

$\tan\theta = \sqrt[3]{\frac{3}{2}}$ $\qquad$ $\left(\frac{3}{2}\right)^{\frac{1}{3}}$

$\theta = \tan^{-1}\left(\sqrt[3]{\frac{3}{2}}\right)$

$\theta \doteq 48.9°$

$l = \frac{3}{\sin 48.9°} + \frac{2}{\cos 48.9°}$

$\doteq 7.02$

$\therefore$ the length of the longest thin straight rod is about 7.02 m or 702 cm

HW: Worksheet on Applications of Trig Functions – Optimization